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A KIND OF GEOMETRY INEQUALITIES INVOLVING AN INTERIOR POINT OF A TRIANGLE

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Abstract

In this paper, we establish three geometric inequalities involving an arbitrary point of a triangle. We also propose three related conjectures which have been checked by the computer.

Key words: triangle, interior point, medians, Euler's inequality, Gerretsen's inequality, the fundamental triangle inequality

1. Introduce and main results

In a Chinese paper [1], the author established the following geometric inequality:

Let P be an interior point of a triangle ABC. Let R_1, R_2, R_3 be the distances from P to the vertices A, B, C, respectively, and let r_1, r_2, r_3 be the distances from P to the sides BC, CA, AB, respectively. Denote by a, b, c the side lengths BC, CA, AB, respectively. Then for any real numbers x, y, z the following inequality holds:

$$a\frac{R_1}{r_1}x^2 + b\frac{R_2}{r_2}y^2 + c\frac{R_3}{r_3}z^2 \geq 2(yza + zxb + xyc), \tag{1.1}$$

with equality if and only if $x : y : z = \cos A : \cos B : \cos C$ and P is the circumcenter of triangle ABC.

In the Chapter III of the author's monograph [2], several equivalent forms of inequality (1.1) are given. One of them is

$$aR_1x^2 + bR_2y^2 + cR_3z^2 \geq 2(yzar_1 + zxbr_2 + xycr_3), \tag{1.2}$$

with equality if and only if $x = y = z$ and P is the orthocenter of triangle ABC. We also gave some applications of inequalities (1.1) and (1.2) in [1] and [2].

An obvious corollary of (1.3) is

$$a\frac{R_1}{r_1} + b\frac{R_2}{r_2} + c\frac{R_3}{r_3} \geq 4s, \tag{1.3}$$

where s is the semi-perimeter of triangle ABC. The following two similar inequalities were also obtained from (1.1):

$$(s-a)\frac{R_1}{r_1} + (s-b)\frac{R_2}{r_2} + (s-c)\frac{R_3}{r_3} \geq 2s, \tag{1.4}$$

$$(b+c)\frac{R_1}{r_1} + (c+a)\frac{R_2}{r_2} + (a+b)\frac{R_3}{r_3} \geq 4s. \tag{1.5}$$

Recently, inspired by inequality (1.3), the author first found that the following similar inequality holds:

$$m_a\frac{R_1}{r_1} + m_b\frac{R_2}{r_2} + m_c\frac{R_3}{r_3} \geq 2(m_a + m_b + m_c). \tag{1.6}$$

where m_a, m_b, m_c are the corresponding medians of triangle ABC. Subsequently, we further obtain the following stronger result:

Theorem 1. *For an interior point P of triangle ABC the following inequality holds:*

$$m_a\frac{R_1}{r_1}+m_b\frac{R_2}{r_2}+m_c\frac{R_3}{r_3}\geq\sqrt{12(m_a^2+m_b^2+m_c^2)},\tag{1.7}$$

with equality if and only if triangle ABC is equilateral and P is its center.

By the power mean inequality, we know that inequality (1.7) is stronger than (1.6). In addition, by the well-known identity:

$$m_a^2+m_b^2+m_c^2=\frac{3}{4}(a^2+b^2+c^2),\tag{1.8}$$

we see that inequality (1.7) is equivalent to

$$m_a\frac{R_1}{r_1}+m_b\frac{R_2}{r_2}+m_c\frac{R_3}{r_3}\geq 3\sqrt{a^2+b^2+c^2}.\tag{1.9}$$

For an acute triangle ABC, we have the following inequality (see [3, p.248])

$$a^2+b^2+c^2\geq 4(R+r)^2,\tag{1.10}$$

where R and r are the circumradius and inradius of triangle ABC. This inequality and inequality (1.8) yields that the following inequality (1.11) holds for the acute triangle. We shall further prove that it holds for any triangle.

Theorem 2. *For an interior point P of triangle ABC the following inequality holds:*

$$m_a\frac{R_1}{r_1}+m_b\frac{R_2}{r_2}+m_c\frac{R_3}{r_3}\geq 6(R+r),\tag{1.11}$$

with equality if and only if triangle ABC is equilateral and P is its center.

Motivated by inequality (1.6), we shall prove the following similar result:

Theorem 3. *For an interior point P of triangle ABC the following inequality holds:*

$$m_a^2\frac{R_1}{r_1}+m_b^2\frac{R_2}{r_2}+m_c^2\frac{R_3}{r_3}\geq 2(m_a^2+m_b^2+m_c^2),\tag{1.12}$$

with equality if and only if triangle ABC is equilateral and P is its center.

Identity (1.8) shows that inequality (1.12) is equivalent to

$$m_a^2 \frac{R_1}{r_1} + m_b^2 \frac{R_2}{r_2} + m_c^2 \frac{R_3}{r_3} \geq \frac{3}{2}(a^2 + b^2 + c^2). \tag{1.13}$$

The main purpose of this paper is to prove Theorem 1, 2 and 3. We also propose three related conjectures.

2. **Proof of Theorem 1**

In order to prove Theorem 1, we need several lemmas. In what follows, we shall continuously use the above symbols. In addition, we shall denote $\sum$ by cyclic sums over a triple or multiple triples. For example,

$$\sum b^2c^2 = b^2c^2 + c^2a^2 + a^2b^2,$$
$$\sum m_a^2 = m_a^2 + m_b^2 + m_c^2,$$
$$\sum m_a \frac{R_1}{r_1} = m_a \frac{R_1}{r_1} + m_b \frac{R_2}{r_2} + m_c \frac{R_3}{r_3}.$$

Lemma 1. *For an interior point of triangle ABC the following inequality holds:*

$$2R_1 \sin \frac{A}{2} \geq r_2 + r_3, \tag{2.1}$$

with equality if and only if $r_2 = r_3$.

Inequality (2.1) is easily proved (see the proof of inequality 12.20 in [4]). Two similar relations are also valid.

Lemma 2. *For any positive real numbers x, y, z and u, v, w, the following inequality holds:*

$$\sum \frac{x^3}{u^2} \geq \frac{\left(\sum x\right)^3}{\left(\sum u\right)^2}, \tag{2.2}$$

with equality if and only if $x : y : z = u : v : w$.

Inequality (2.2) is a special case of the Radon inequality (cf. [5, Theorem 65]).

Lemma 3. *In any triangle ABC the following inequality holds:*

$$\sin\frac{B}{2}\sin\frac{C}{2} \leq \frac{s-a}{2a}\left(\frac{s-b}{c}+\frac{s-c}{b}\right), \tag{2.3}$$

with equality if and only if $b = c$.

Proof

Using the following well-known formula in triangle ABC:

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \tag{2.4}$$

we get

$$\sin\frac{B}{2}\sin\frac{C}{2} = \frac{s-a}{a}\sqrt{\frac{(s-b)(s-c)}{bc}}. \tag{2.5}$$

Then inequality (2.3) follows immediately by using the simplest arithmetic-geometric mean inequality. Also, the equality occurs only when $(s-b)/c = (s-c)/b$, i.e., $b = c$. Lemma 3 is proved.

Lemma 4. *In any triangle ABC the following inequality holds:*

$$m_a \leq \frac{8S^2 + bc(b-c)^2}{4aS}, \tag{2.6}$$

where S is the area of triangle ABC. Equality in (2.6) *holds if and only if $b = c$.*

Inequality (2.6) is one of the equivalent conclusion of Theorem 1.1 from [6].

Lemma 5. *In any triangle ABC the following inequality holds:*

$$m_a m_b m_c \geq \frac{1}{8R}\sum b^2c^2, \tag{2.7}$$

with equality if and only if triangle ABC is isosceles.

Inequality (2.7) and its equivalent forms have been used by the author in [7] and [8].

The following two lemmas give some identities for the sums $\sum a^n$ and $\sum(bc)^n$ in term of R, r and s.

Lemma 6. *In the triangle ABC, let $p_n = \sum a^n$ (n being natural number). Then the following identities hold:*

$$p_1 = 2s, \tag{2.8}$$

$$p_2 = 2s^2 - 8Rr - 2r^2, \tag{2.9}$$

$$p_3 = 2s^3 - (12Rr + 6r^2)s, \tag{2.10}$$

$$p_4 = 2s^4 - 4(4R + 3r)rs^2 + 2(4R + r)^2 r^2, \tag{2.11}$$

$$p_5 = 2s^5 - 20(R + r)rs^3 + 10(2R + r)(4R + r)r^2 s, \tag{2.12}$$

$$\begin{aligned} p_6 = {} & 2s^6 - 6(4R + 5r)rs^4 + 6(24R^2 + 24Rr + 5r^2)r^2 s^2 \\ & - 2(4R + r)^3 r^3, \end{aligned} \tag{2.13}$$

$$\begin{aligned} p_7 = {} & 2s^7 - 14(2R + 3r)rs^5 + 14(16R^2 + 20Rr + 5r^2)r^2 s^3 \\ & - 14(2R + r)(4R + r)^2 r^3 s, \end{aligned} \tag{2.14}$$

$$\begin{aligned} p_8 = {} & 2s^8 - 8(4R + 7r)rs^6 + 20(16R^2 + 24Rr + 7r^2)r^2 s^4 \\ & - 8(4R + r)(32R^2 + 32Rr + 7r^2)r^3 s^2 + 2(4R + r)^4 r^4, \end{aligned} \tag{2.15}$$

$$\begin{aligned} p_9 = {} & 2s^9 - 36(R + 2r)rs^7 + 36(12R^2 + 21Rr + 7r^2)r^2 s^5 \\ & - 12(160R^3 + 240R^2 r + 105Rr^2 + 14r^3)r^3 s^3 \\ & + 18(2R + r)(4R + r)^3 r^4 s, \end{aligned} \tag{2.16}$$

$$\begin{aligned} p_{10} = {} & 2s^{10} - 10(4R + 9r)rs^8 + 140(2R + 3r)(2R + r)r^2 s^6 \\ & - 20(160R^3 + 280R^2 r + 140Rr^2 + 21r^3)r^3 s^4 \\ & + 10(40R^2 + 40Rr + 9r^2)(4R + r)^2 r^4 s^2 - 2(4R + r)^5 r^5, \end{aligned} \tag{2.17}$$

$$\begin{aligned} p_{11} = {} & 2s^{11} - 22(2R + 5r)rs^9 + 44(16R^2 + 36Rr + 15r^2)r^2 s^7 \\ & - 308(2R + r)(8R^2 + 12Rr + 3r^2)r^3 s^5 \\ & + 22(4R + r)(160R^3 + 240R^2 r + 108Rr^2 + 15r^3)r^4 s^3 \\ & - 22(2R + r)(4R + r)^4 r^5 s, \end{aligned} \tag{2.18}$$

$$\begin{aligned} p_{12} = {} & 2s^{12} - 12(4R + 11r)rs^{10} + 18(48R^2 + 120Rr + 55r^2)r^2 s^8 \\ & - 56(128R^3 + 288R^2 r + 180Rr^2 + 33r^3)r^3 s^6 + 6(4480R^4 \\ & + 8960R^3 r + 6048R^2 r^2 + 1680Rr^3 + 165r^4)r^4 s^4 \\ & - 12(48R^2 + 48Rr + 11r^2)(4R + r)^3 r^5 s^2 + 2(4R + r)^6 r^6, \end{aligned} \tag{2.19}$$

$$
\begin{aligned}
p_{13} =& 2s^{13} - 52(R+3r)rs^{11} + 130(8R^2+22Rr+11r^2)r^2s^9 \\
& - 312(32R^3+80R^2r+55Rr^2+11r^3)r^3s^7 + 26(1792R^4 \\
& + 4032R^3r + 3024R^2r^2 + 924Rr^3 + 99r^4)r^4s^5 \\
& - 52(112R^3+168R^2r+77Rr^2+11r^3)(4R+r)^2r^5s^3 \\
& + 26(2R+r)(4R+r)^5r^6s, \qquad (2.20)\\
p_{14} =& 2s^{14} - 14(4R+13r)rs^{12} + 154(8R^2+24Rr+13r^2)r^2s^{10} \\
& - 42(320R^3+880R^2r+660Rr^2+143r^3)r^3s^8 \\
& + 42(1792R^4+4480R^3r+3696R^2r^2+1232Rr^3 \\
& + 143r^4)r^4s^6 - 14(4R+r)(3584R^4+7168R^3r \\
& + 4928R^2r^2 + 1408Rr^3 + 143r^4)r^5s^4 \\
& + 14(56R^2+56Rr+13r^2)(4R+r)^4r^6s^2 \\
& - 2(4R+r)^7r^7, \qquad (2.21)\\
p_{16} =& 2s^{16} - 16(4R+15r)rs^{14} + 104(16R^2+56Rr+35r^2)r^2s^{12} \\
& - 176(128R^3+416R^2r+364Rr^2+91r^3)r^3s^{10} \\
& + 132(1280R^4+3840R^3r+3744R^2r^2+1456Rr^3 \\
& + 195r^4)r^4s^8 - 16(43008R^5+118272R^4r+118272R^3r^2 \\
& + 54912R^2r^3 + 12012Rr^4 + 1001r^5)r^5s^6 + 8(10752R^4 \\
& + 21504R^3r + 14976R^2r^2 + 4368Rr^3 + 455r^4)(4R+r)^2r^6s^4 \\
& - 16(8R+5r)(8R+3r)(4R+r)^5r^7s^2 \\
& + 2(4R+r)^8r^8. \qquad (2.22)
\end{aligned}
$$

Lemma 7. *In the triangle ABC, let $q_n = \sum(bc)^n$ (n being natural number). Then the following identities hold:*

$$
\begin{aligned}
q_1 =& s^2 + 4Rr + r^2, \qquad (2.23)\\
q_2 =& s^4 - 2(4R-r)rs^2 + (4R+r)^2r^2, \qquad (2.24)\\
q_3 =& s^6 - 3(4R-r)rs^4 + 3r^4s^2 + (4R+r)^3r^3, \qquad (2.25)\\
q_5 =& s^{10} - 5(4R-r)rs^8 + 10(8R^2-4Rr+r^2)r^2s^6 \\
& + 10r^6s^4 + 5(4R+r)^2r^6s^2 + (4R+r)^5r^5, \qquad (2.26)
\end{aligned}
$$

$$\begin{aligned}
q_6 =& s^{12} - 6(4R-r)rs^{10} + 3(48R^2 - 24Rr + 5r^2)r^2s^8 \\
&- 4(32R^3 - 24R^2r + 12Rr^2 - 5r^3)r^3s^6 \\
&+ 3(16R+5r)r^7s^4 + 6(4R+r)^3r^7s^2 \\
&+ (4R+r)^6r^6,
\end{aligned} \tag{2.27}$$

$$\begin{aligned}
q_8 =& s^{16} - 8(4R-r)rs^{14} + 4(80R^2 - 40Rr + 7r^2)r^2s^{12} \\
&- 8(128R^3 - 96R^2r + 36Rr^2 - 7r^3)r^3s^{10} \\
&+ 2(256R^4 - 256R^3r + 160R^2r^2 - 80Rr^3 + 35r^4)r^4s^8 \\
&+ 8(20R+7r)r^9s^6 + 4(16R+7r)(4R+r)^2r^9s^4 \\
&+ 8(4R+r)^5r^9s^2 + (4R+r)^8r^8.
\end{aligned} \tag{2.28}$$

Remark 1. *Identities* (2.9), (2.10), (2.11) *and* (2.23) *in Lemma 6 and 7 are well known. The proofs of the other identities can be found in [9] and [10]. Identities* (2.20)-(2.24),(2.26) *and* (2.28) *will not be used in the proof of Theorem 1, but will be used in the proof of Theorem 2.*

We are now ready to prove Theorem 1.

Proof

By Lemma 1, to prove inequality (1.7) we need to show that

$$\sum m_a \frac{r_2 + r_3}{2r_1 \sin\dfrac{A}{2}} \ge \sqrt{12 \sum m_a^2},$$

which is equivalent to

$$\sum \left(\frac{m_b r_3}{r_2 \sin\dfrac{B}{2}} + \frac{m_c r_2}{r_3 \sin\dfrac{C}{2}} \right) \ge 2\sqrt{12 \sum m_a^2}. \tag{2.29}$$

Making use of the simplest arithmetic-geometric mean inequality, we have

$$\frac{m_b r_3}{r_2 \sin\dfrac{B}{2}} + \frac{m_c r_2}{r_3 \sin\dfrac{C}{2}} \ge 2\sqrt{\frac{m_b m_c}{\sin\dfrac{B}{2}\sin\dfrac{C}{2}}}.$$

Thus, note that the previous identity (1.8), we only need to prove

$$\sum \sqrt{\frac{m_b m_c}{\sin \frac{B}{2} \sin \frac{C}{2}}} \geq 3\sqrt{\sum a^2}. \tag{2.30}$$

Denoting by r_a, r_b, r_c the radii of escribed circle of triangle ABC. In Lemma 2, we set $x = r_a, y = r_b, z = r_c$ and

$$u = \sqrt{\frac{m_b m_c}{\sin \frac{B}{2} \sin \frac{C}{2}}}, v = \sqrt{\frac{m_c m_a}{\sin \frac{C}{2} \sin \frac{A}{2}}}, w = \sqrt{\frac{m_a m_b}{\sin \frac{A}{2} \sin \frac{B}{2}}},$$

then it follows that

$$\left(\sum \sqrt{\frac{m_b m_c}{\sin \frac{B}{2} \sin \frac{C}{2}}}\right)^2 \sum \frac{r_a^3}{m_b m_c} \sin \frac{B}{2} \sin \frac{C}{2} \geq \left(\sum r_a\right)^3. \tag{2.31}$$

Consequently, to prove inequality (2.30) it remains to show that

$$\left(\sum r_a\right)^3 \geq 9 \sum \frac{r_a^3}{m_b m_c} \sin \frac{B}{2} \sin \frac{C}{2} \sum a^2. \tag{2.32}$$

Since

$$r_a = \frac{S}{s-a}, \tag{2.33}$$

using Lemma 3-5, we have

$$\begin{aligned}
&\sum \frac{r_a^3}{m_b m_c} \sin \frac{B}{2} \sin \frac{C}{2} \\
&= \frac{S^3}{m_a m_b m_c} \sum \frac{m_a}{(s-a)^3} \sin \frac{B}{2} \sin \frac{C}{2} \\
&\leq \frac{8RS^3}{\sum b^2 c^2} \sum \frac{1}{(s-a)^3} \cdot \frac{8S^2 + bc(b-c)^2}{4aS} \cdot \frac{s-a}{2a} \left(\frac{s-b}{c} + \frac{s-c}{b}\right) \\
&= \frac{RS^2}{\sum b^2 c^2} \sum \frac{8S^2 + bc(b-c)^2}{a^2 (s-a)^2} \left(\frac{s-b}{c} + \frac{s-c}{b}\right).
\end{aligned}$$

Thus, to prove inequality (2.32) we need to show that

$$\left(\sum r_a\right)^3 \geq \frac{9RS^2}{\sum b^2c^2} \sum \frac{8S^2 + bc(b-c)^2}{a^2(s-a)^2} \left(\frac{s-b}{c} + \frac{s-c}{b}\right) \sum a^2. \tag{2.34}$$

Using the known identity:

$$abc = 4SR, \tag{2.35}$$

we easily know that (2.34) is equivalent to

$$\begin{aligned} &16R\left(\sum r_a\right)^3 \sum b^2c^2 \prod (s-a)^2 \\ &\geq 9\sum a^2 \sum bc(s-b)^2(s-c)^2[8S^2 + bc(b-c)^2][(s-b)b + (s-c)c], \end{aligned} \tag{2.36}$$

where $\prod$ denotes cyclic product. Again, using the following known identities

$$\sum r_a = 4R + r, \tag{2.37}$$

$$\prod (s-a) = sr^2, \tag{2.38}$$

one sees that inequality (2.36) is equivalent to

$$A_0 \equiv 16R(4R+r)^3 r^4 s^2 \sum b^2c^2 - 9B_0 \sum a^2 \geq 0, \tag{2.39}$$

where

$$B_0 = \sum bc(s-b)^2(s-c)^2[8S^2 + bc(b-c)^2][(s-b)b + (s-c)c].$$

Next, we shall compute B_0 in terms of R, r and s. With the help of MAPLE software, using $s = (a+b+c)/2$ and Heron's formula

$$S = \sqrt{s(s-a)(s-b)(s-c)} \tag{2.40}$$

one can easily obtain the following identity:

$$\begin{aligned}64B_0 =& -5b^8ca^3 + 8b^6c^3a^3 - 104b^5c^5a^2 + 90b^4c^6a^2 + 8b^3c^6a^3\\&- 76b^3c^7a^2 + 8b^3c^3a^6 + 90b^2c^6a^4 + 46b^2c^8a^2 + 90b^4c^2a^6\\&- 180b^4c^4a^4 + 44b^3c^4a^5 - 76b^2c^7a^3 + 44b^4c^5a^3 - 76b^7c^3a^2\\&+ 29b^9c^3 - 64b^8c^4 + 98b^7c^5 - 112b^6c^6 + 98b^5c^7 - 8b^2c^{10}\\&- 64b^4c^8 - 8b^{10}c^2 + b^{11}c + 44b^4c^3a^5 + 90b^2c^4a^6 - 5b^3c^8a\\&+ 46b^2c^2a^8 + 44b^3c^5a^4 + 20b^7c^4a - 104b^2c^5a^5 + 44b^5c^3a^4\\&+ 20b^4c^7a + 90b^6c^2a^4 + bc^{11} + 20b^4ca^7 - 14b^5ca^6\\&- 14b^6c^5a - 5b^8c^3a - 104b^5c^2a^5 - 2b^{10}ca - 14b^5c^6a\\&- 14bc^5a^6 - 5bc^8a^3 - 14b^6ca^5 + 20b^7ca^4 - 76b^7c^2a^3\\&+ 44b^5c^4a^3 - 14bc^6a^5 + 46b^8c^2a^2 - 2bc^{10}a + 90b^6c^4a^2\\&+ 29b^3c^9 - 8c^2a^{10} + 29c^3a^9 + 20bc^7a^4 + 20bc^4a^7\\&- 5b^3ca^8 - 5bc^3a^8 - 64c^8a^4 + 29c^9a^3 + ca^{11} - 8c^{10}a^2\\&- 112c^6a^6 + 98c^7a^5 - 64c^4a^8 + 98c^5a^7 - 76c^2a^7b^3\\&- 76c^3a^7b^2 + 29a^9b^3 + c^{11}a - 64a^8b^4 + 98a^7b^5\\&- 2ca^{10}b - 8a^{10}b^2 + a^{11}b - 64a^4b^8 - 112a^6b^6\\&+ 98a^5b^7 + 29a^3b^9 - 8a^2b^{10} + ab^{11}.\end{aligned}\tag{2.41}$$

We set $d = abc$, then it is not difficult to obtain the following identity:

$$\begin{aligned}64B_0 =& -180d^4 + (44p_1p_2 - 36p_3)d^3 + (32p_6 - 76p_1p_5 + 90p_2p_4\\&- 104q_3)d^2 + (20p_3p_6 - 3p_9 - 14p_4p_5 - 5p_2p_7)d - 56p_{12}\\&+ 98p_5p_7 + 29p_3p_9 - 112q_6 - 64p_4p_8 - 8p_2p_{10} + p_1p_{11}.\end{aligned}\tag{2.42}$$

With the help of MAPLE software, using some related identities given in Lemma 6, 7 and the following identity:

$$abc = 4Rrs \tag{2.43}$$

one can easily obtain the following identity:

$$\begin{aligned}B_0 =& 8r^5\left[(-40R^3 - 100R^2r - 14Rr^2 + r^3)s^4\right.\\&\left.+2(2R + 11r)(4R + r)^3Rs^2 - (4R + r)^6r\right].\end{aligned}\tag{2.44}$$

Now, using identities (2.10),(2.25) and (2.44), we easily get

$$A_0 = 16r^4 C_0, \tag{2.45}$$

where

$$\begin{aligned}C_0 =&(64R^4 + 408R^3r + 912R^2r^2 + 127Rr^3 - 9r^4)s^6\\ &- (4R + r)(704R^4 + 3848R^3r + 2512R^2r^2 + 322Rr^3\\ &- 9r^4)rs^4 + (184R^2 + 271Rr + 9r^2)(4R + r)^4r^2s^2\\ &- 9(4R + r)^7r^3.\end{aligned}$$

Consequently, for proving inequality (2.39) we have to prove that

$$C_0 \geq 0. \tag{2.46}$$

We recall that for any triangle we have Euler's inequality

$$e \equiv R - 2r \geq 0 \tag{2.47}$$

and Gerretsen's inequality (cf. [3], [4] and [11]):

$$x_0 \equiv s^2 - 16Rr + 5r^2 \geq 0. \tag{2.48}$$

Based on these two inequality, after analysis we obtain the following identity:

$$C_0 = C_1x_0^3 + rC_2x_0^2 + 4r^2C_3, \tag{2.49}$$

where

$$\begin{aligned}C_1 =&64R^4 + 408R^3r + 912R^2r^2 + 127Rr^3 - 9r^4,\\ C_2 =&256R^5 + 2528R^4r + 23760R^3r^2 - 11384R^2r^3\\ &- 2623Rr^4 + 144r^5,\\ C_3 =&(1536R^6 - 21952R^5r + 78752R^4r^2 - 65250R^3r^3\\ &+ 8690R^2r^4 + 4352Rr^5 - 189r^6)s^2 + (12288R^7\\ &+ 140288R^6r - 908096R^5r^2 + 1017944R^4r^3\\ &- 311314R^3r^4 - 25502R^2r^5 + 15422Rr^6 - 621r^7)r.\end{aligned}$$

By Euler's inequality one sees that $C_1 > 0$ and $C_2 > 0$. Since $x_0 \geq 0$, to prove $C_0 \geq 0$ it remains to show $C_3 \geq 0$.

For any triangle we have the following two inequalities (due to Yang, cf. [11]):

$$i_0 \equiv (R-r)s^2 - r(16R^2 - 20Rr + 3r^2) \geq 0, \tag{2.50}$$

$$j_0 \equiv 4R^3 - 2Rr^2 - r^3 - (R-r)s^2 \geq 0, \tag{2.51}$$

which are stronger than Gerretsen's inequalities (2.48) and (3.22) below respectively. We easily check the following identity:

$$(e+r)C_3 = C_4 i_0 + C_5 j_0 + C_6, \tag{2.52}$$

where

$$\begin{aligned} C_4 =& 1536e^6 + 119718e^2r^4 + 314928er^5 + 177147r^6, \\ C_5 =& 2re^3(1760e^2 + 24304er + 33777r^2), \\ C_6 =& er(22784e^7 + 113280e^6r + 562616e^5r^2 + 1650152e^4r^3 \\ &+ 2126250e^3r^4 + 1134486e^2r^5 + 284310er^6 + 59049r^7). \end{aligned}$$

By Euler's inequality $e \geq 0$ we have $C_4 > 0, C_5 \geq 0$ and $C_6 \geq 0$. And then from (2.52) we deduce $C_3 \geq 0$. Inequalities (2.39), (2.30) and (1.7) are proved. Moreover, it is easy to determine that equality in (1.7) holds only when triangle ABC is equilateral and P is its center. This completes the proof of Theorem 1.

3. Proof of Theorem 2

In this section, we shall prove Theorem 2. We first give two lemmas.

Lemma 8. *In any triangle ABC the following inequality holds:*

$$\sin\frac{B}{2}\sin\frac{C}{2} \leq \frac{s-a}{b+c}, \tag{3.1}$$

with equality if and only if $b = c$.

Proof

From the previous identity (2.5), using formula (2.4) and the well-known inequality:

$$\sin\frac{A}{2} \leq \frac{a}{b+c}, \tag{3.2}$$

we immediately obtain inequality (3.1). Also, the equality condition of (3.1) is the same as (3.2), i.e., $b = c$. The proof is completed.

Remark 2. *It is easy to prove that inequality* (3.1) *is weaker than the previous inequality* (2.3).

Lemma 9. *In any triangle ABC the following identity holds:*

$$16(m_a m_b m_c)^2 = s^6 - 3(4R - 11r)rs^4 - 3(20R^2 + 40Rr + 11r^2)r^2s^2 - (4R + r)^3r^3. \tag{3.3}$$

Proof

Using the known median formula

$$4m_a^2 = 2b^2 + 2c^2 - a^2 \tag{3.4}$$

we easily get

$$64(m_a m_b m_c)^2 = 6\sum a^2 \sum a^4 - 10\sum a^6 + 3(abc)^2. \tag{3.5}$$

Then identity (3.3) follows immediately by using the previous identities (2.9), (2.11), (2.13) and (2.43). The poof is completed.

We now prove Theorem 2.

Proof

By Lemma 1, to prove inequality (1.11) we need to show

$$\sum m_a \frac{r_2 + r_3}{r_1 \sin\dfrac{A}{2}} \geq 12(R + r),$$

which is equivalent to

$$\sum \left(\frac{r_3 m_b}{r_2 \sin\dfrac{B}{2}} + \frac{r_2 m_c}{r_3 \sin\dfrac{C}{2}} \right) \geq 12(R + r). \tag{3.6}$$

By the arithmetic-geometric mean inequality, we only need to prove that

$$\sum \sqrt{\frac{m_b m_c}{\sin\dfrac{B}{2} \sin\dfrac{C}{2}}} \geq 6(R + r). \tag{3.7}$$

Denoting by h_a, h_b, h_c by the corresponding altitudes of triangle ABC and denoting by r_a, r_b, r_c the radii of escribed circle of triangle ABC. In inequality given in Lemma 2, we take

$$x = h_a + r_a,\ y = h_b + r_b,\ z = h_c + r_c,$$
$$u = \sqrt{\frac{m_b m_c}{\sin\frac{B}{2}\sin\frac{C}{2}}},\ v = \sqrt{\frac{m_c m_a}{\sin\frac{C}{2}\sin\frac{A}{2}}},\ w = \sqrt{\frac{m_a m_b}{\sin\frac{A}{2}\sin\frac{B}{2}}}.$$

Then it follows that

$$\left(\sum\sqrt{\frac{m_b m_c}{\sin\frac{B}{2}\sin\frac{C}{2}}}\right)^2 \sum\frac{(h_a+r_a)^3}{m_b m_c}\sin\frac{B}{2}\sin\frac{C}{2} \geq \left[\sum(h_a+r_a)\right]^3. \tag{3.8}$$

Thus, to prove inequality (3.7) we only need to show

$$\left[\sum(h_a+r_a)\right]^3 \geq 36(R+r)^2\sum\frac{(h_a+r_a)^3}{m_b m_c}\sin\frac{B}{2}\sin\frac{C}{2}. \tag{3.9}$$

Using $h_a = 2S/a$ and $r_a = S/(s-a)$ we get

$$h_a + r_a = \frac{(b+c)S}{a(s-a)}. \tag{3.10}$$

And, then by Lemma 8 we have

$$(h_a+r_a)^3\sin\frac{B}{2}\sin\frac{C}{2} \leq \frac{(b+c)^2S^3}{a^3(s-a)^2}. \tag{3.11}$$

Therefore, to prove inequality (3.9) it remains to prove that

$$\left(\sum h_a + \sum r_a\right)^3 \geq 36(R+r)^2S^3\sum\frac{(b+c)^2}{m_b m_c a^3(s-a)^2}. \tag{3.12}$$

Note that

$$2m_b m_c \leq \frac{c(s-c)}{b(s-b)}m_b^2 + \frac{b(s-b)}{c(s-c)}m_c^2. \tag{3.13}$$

To prove inequality (3.12) we need to show that

$$\begin{aligned}&\left(\sum h_a+\sum r_a\right)^3\\&\geq 18(R+r)^2S^3\sum\left[\frac{c(s-c)}{b(s-b)}m_b^2+\frac{b(s-b)}{c(s-c)}m_c^2\right]\frac{(b+c)^2}{m_b^2m_c^2a^3(s-a)^2}.\end{aligned}$$

Multiplying both sides by $8(m_am_bm_c)^2R^3$, and using $2Rh_a=bc$ and identity $abc=4SR$, one can obtain the following equivalent inequality:

$$\begin{aligned}&(m_am_bm_c)^2\left(\sum bc+8R^2+2Rr\right)^3\\&\geq 144(R+r)^2R^3S^3\sum\frac{(b+c)^2m_a^2}{a^3(s-a)^2}\left[\frac{c(s-c)}{b(s-b)}m_b^2+\frac{b(s-b)}{c(s-c)}m_c^2\right],\end{aligned}$$

Again, multiplying both sides by $16\prod(s-a)^2$ and using identity $abc=4SR$, the above inequality becomes

$$\begin{aligned}&16(m_am_bm_c)^2\left(\sum bc+8R^2+2Rr\right)^3\prod(s-a)^2\\&\geq 36(R+r)^2\\&\cdot\sum(s-b)(s-c)(b+c)^2b^2c^2m_a^2\left[b^2(s-b)^2m_c^2+c^2(s-c)^2m_b^2\right].\end{aligned}\tag{3.14}$$

Using the previous identities (2.23) and (2.38), we further know that the above inequality is equivalent to

$$D_0\equiv 16r^2s^4\left(s^2+8R^2+6Rr+r^2\right)^3(m_am_bm_c)^2-36(R+r)^2E_0\geq 0,\tag{3.15}$$

where

$$E_0=\sum(s-b)(s-c)(b+c)^2b^2c^2m_a^2\left[b^2(s-b)^2m_c^2+c^2(s-c)^2m_b^2\right].$$

In order to prove inequality (3.15), we next first compute E_0 in terms of R,r and s. With the help of MAPLE software, using formula (3.4)

and $s=(a+b+c)/2$, one can easily obtain the following identity:

$$\begin{aligned}128E_0 =&12b^7c^3a^6+2b^5c^5a^6+12b^6c^3a^7+2b^9c^2a^5+6b^7c^5a^4\\
&-2b^9c^3a^4+5b^6c^8a^2-12b^{10}c^3a^3+12b^7c^6a^3+6b^7c^4a^5\\
&+6b^5c^4a^7+2b^8c^4a^4-12b^7c^7a^2+2b^9c^5a^2+2b^6c^5a^5\\
&+12b^6c^7a^3+10b^4c^6a^6+10b^6c^6a^4-8b^5c^{11}+2b^{10}c^6\\
&+2b^5c^8a^3+2b^8c^5a^3+6b^5c^7a^4-b^{10}c^4a^2-b^4c^{10}a^2\\
&+6b^4c^7a^5+2b^5c^9a^2-2b^4c^9a^3-2b^9c^4a^3+2b^4c^8a^4\\
&-2b^{12}c^2a^2-2b^2c^{12}a^2+10b^6c^4a^6+5b^6c^2a^8-12b^2c^7a^7\\
&-12b^7c^2a^7-b^{10}c^2a^4+5b^8c^2a^6+5b^2c^6a^8+5b^2c^8a^6\\
&-b^2c^{10}a^4+2b^2c^9a^5+4b^{13}c^2a+4b^2c^{13}a-4b^4c^{11}a\\
&-4b^3c^{12}a+8b^8c^7a+8b^7c^8a-4b^{12}c^3a+4b^{10}c^5a\\
&-8b^6c^9a-4b^{11}c^4a+4b^5c^{10}a-8b^9c^6a-2b^3c^9a^4\\
&+12b^3c^7a^6+12b^3c^6a^7+2b^5c^3a^8+4b^7c^9-4b^8c^8\\
&+2b^{12}c^4+2b^3c^5a^8+6b^4c^5a^7+2b^4c^4a^8-2b^{14}c^2\\
&-2b^2c^{14}-2a^2c^{14}+2a^4c^{12}+4a^9c^7+2a^{10}c^6-4a^8c^8\\
&-12b^3c^{10}a^3+5b^8c^6a^2+2a^6c^{10}+4a^{13}c^3-2a^{14}c^2\\
&+4b^3c^{13}+2b^4c^{12}+4b^9c^7+4b^{13}c^3-8b^{11}c^5+4a^3c^{13}\\
&+4a^7c^9-8a^5c^{11}-8a^{11}c^5+2a^{12}c^4+4a^7b^9+2a^5c^8b^3\\
&-2a^9c^4b^3-2a^9c^3b^4+2a^5c^3b^8+2b^6c^{10}-2a^{12}c^2b^2\\
&+4a^2c^{13}b+4a^{13}c^2b-4a^3c^{12}b-8a^6c^9b+8a^8c^7b\\
&+4a^{10}c^5b-8a^9c^6b-12a^{10}c^3b^3-a^{10}c^4b^2+2a^9c^5b^2\\
&+8a^7c^8b+4a^5c^{10}b-4a^4c^{11}b+2a^{10}b^6+4a^9b^7\\
&+2a^6b^{10}-2a^2b^{14}-4a^8b^8+4a^3b^{13}-8a^{11}b^5\\
&+2a^{12}b^4+4a^{13}b^3-a^{10}c^2b^4-4a^{12}c^3b-4a^{11}c^4b\\
&+2a^9c^2b^5+2a^4b^{12}-2a^{14}b^2-8a^5b^{11}+4a^2b^{13}c\\
&-4a^3b^{12}c+8a^8b^7c-8a^9b^6c+4a^{10}b^5c+4a^5b^{10}c\\
&+8a^7b^8c-8a^6b^9c-4a^4b^{11}c+4a^{13}b^2c-4a^{11}b^4c\\
&-4a^{12}b^3c+2b^5c^6a^5.\end{aligned}\tag{3.16}$$

We continuously use symbols in Lemma 6 and 7. By calculation we obtain identity:

$$\begin{aligned}128E_0 =&2p_1d^5 + (6p_1p_3 - 4p_4 + 10q_2)d^4 + (2p_2p_5 - 24p_7\\ &+ 12p_3p_4 - 2p_1p_6)d^3 + (5p_4p_6 + 2p_3p_7 - 8p_{10}\\ &- 12q_5 - p_2p_8)d^2 + (4p_1p_{12} + 4p_4p_9 - 8p_5p_8\\ &+ 8p_6p_7 - 4p_2p_{11} - 4p_3p_{10})d + 2p_4p_{12} + 4p_3p_{13}\\ &+ 2p_6p_{10} - 4q_8 - 2p_{16} - 8p_5p_{11} + 4p_7p_9 - 2p_2p_{14}. \qquad (3.17)\end{aligned}$$

With the help of MAPLE software, using $d = abc = 4Rrs$ and related identities given in Lemma 6 and 7, we further obtain the following identity:

$$E_0 = 2s^2r^4F_0, \qquad (3.18)$$

where

$$\begin{aligned}F_0 =& - s^{10} + (8R^2 + 48Rr + 13r^2)s^8 - 2(124R^3 + 312R^2r\\ &+ 76Rr^2 - 7r^3)rs^6 + 2(80R^5 + 1312R^4r + 1584R^3r^2\\ &+ 574R^2r^3 + 44Rr^4 - 7r^5)rs^4 - (176R^4 + 472R^3r\\ &+ 328R^2r^2 + 96Rr^3 + 13r^4)(4R + r)^2r^2s^2\\ &+ (2R + r)^2(4R + r)^5r^3.\end{aligned}$$

Finally, by using identities (3.3) and (3.18), we can obtain the following identity:

$$D_0 = r^4s^2G_0, \qquad (3.19)$$

where

$$\begin{aligned}G_0 =&s^{12} + (96R^2 + 150Rr + 108r^2)s^{10} - (384R^4 + 4608R^3r\\ &+ 7752R^2r^2 + 4854Rr^3 + 867r^4)s^8 + (512R^6 + 16704R^5r\\ &+ 83136R^4r^2 + 122768R^3r^3 + 67476R^2r^4 + 9132Rr^5\\ &- 1008r^6)s^6 - 3(5888R^7 + 73472R^6r + 211328R^5r^2\\ &+ 250512R^4r^3 + 138808R^3r^4 + 33404R^2r^5 + 1780Rr^6\\ &- 313r^7)rs^4 + 6(1792R^6 + 8560R^5r + 15496R^4r^2\end{aligned}$$

$$+ 13408R^3r^3 + 5938R^2r^4 + 1381Rr^5 + 150r^6)(4R+r)^2r^2s^2$$
$$- (80R^2 + 150Rr + 73r^2)(2R+r)^2(4R+r)^5r^3.$$

Therefore, for proving inequality (3.15) we need to prove that

$$G_0 \geq 0. \tag{3.20}$$

We now recall that for any triangle ABC the following fundamental triangle inequality

$$t_0 \equiv -s^4 + (4R^2 + 20Rr - 2r^2)s^2 - r(4R+r)^3 \geq 0 \tag{3.21}$$

holds (for the proofs, see for example, [3] and [11]). Moreover, from this inequality one can obtain the previous Gerretsen inequality (2.48) and another Gerretsen inequality (see [3]):

$$y_0 \equiv 4R^2 + 4Rr + 3r^2 - s^2 \geq 0. \tag{3.22}$$

Based on inequalities (3.21), (3.22), (2.48) and Euler's inequality (2.47), after analysis we obtain the following identity:

$$G_0 = (y_0x_0^3 + m_1)t_0 + H_0x_0^3 + m_2x_0^2 + (R-2r)(m_3x_0 + m_4), \tag{3.23}$$

where

$$\begin{aligned}
m_1 =& 72r^4(2862R^4 + 15795R^3r + 5554R^2r^2 + 3328r^4),\\
m_2 =& 576Rr(12R^6 - 28R^5r + 571R^4r^2 + 1566r^6),\\
m_3 =& 144r^3(2240R^6 + 9540R^5r - 8844R^4r^2 + 18359R^3r^3\\
& - 9790R^2r^4 + 6136Rr^5 + 4336r^6),\\
m_4 =& 576r^5(5700R^6 + 12915R^5r + 5238R^4r^2 - 32452R^3r^3\\
& + 9116R^2r^4 - 3088Rr^5 - 2704r^6),\\
H_0 =& (104R^2 + 222Rr + 94r^2)s^4 - (400R^4 + 160R^3r + 588R^2r^2\\
& + 2944Rr^3 + 2332r^4)s^2 + 512R^6 - 1472R^5r + 15616R^4r^2\\
& - 576R^3r^3 - 51152R^2r^4 - 28814Rr^5 + 26950r^6.
\end{aligned}$$

Note that

$$12R^6 - 28R^5r + 571R^4r^2 = R^4(12R^2 - 28Rr + 571r^2) > 0,$$

so that $m_2 > 0$. And, by Euler's inequality $R \geq 2r$ we have

$$9540R^5r - 8844R^4r^2 = 12R^4r(795R - 737r) > 0$$

and

$$18359R^3r^3 - 9790R^2r^4 = 11R^2r^3(1669R - 890r) > 0,$$

so that $m_3 > 0$. In addition, it is easy to check that

$$\begin{aligned}&5700R^6 + 12915R^5r + 5238R^4r^2 - 32452R^3r^3 + 9116R^2r^4\\&- 3088Rr^5 - 2704r^6\\&= 5700e^6 + 81315e^5r + 476388e^4r^2 + 1438052e^3r^3\\&+ 2341316e^2r^4 + 1939168er^5 + 629856r^6.\end{aligned} \tag{3.24}$$

Thus $m_4 > 0$ follows from Euler's inequality $e \geq 0$. Therefore, from identity (3.23) by the fundamental triangle inequality $t_0 \geq 0$, Gerretsen's inequalities $x_0 \geq 0$ and $y_0 \geq 0$, it remains to prove the strict inequality $H_0 > 0$, i.e.,

$$H_0 \equiv a_0s^4 + b_0s^2 + c_0 > 0, \tag{3.25}$$

where

$$\begin{aligned}a_0 =&104R^2 + 222Rr + 94r^2,\\b_0 =&-(400R^4 + 160R^3r + 588R^2r^2 + 2944Rr^3 + 2332r^4),\\c_0 =&512R^6 - 1472R^5r + 15616R^4r^2 - 576R^3r^3 - 51152R^2r^4\\&- 28814Rr^5 + 26950r^6.\end{aligned}$$

We set

$$\begin{aligned}I_0 =&(104R^2 + 222Rr + 94r^2)s^2 - 400R^4 + 1504R^3r + 2444R^2r^2\\&- 2550Rr^3 - 2802r^4.\end{aligned}$$

Next, we consider two cases to complete the proof of inequality (3.25).

Case 1. $I_0 > 0$.

It is easy to check that

$$H_0 = x_0I_0 + J_0, \tag{3.26}$$

where

$$\begin{aligned}J_0 =&512R^6 - 7872R^5r + 41680R^4r^2 + 31008R^3r^3\\&- 104172R^2r^4 - 60896Rr^5 + 40960r^6.\end{aligned}$$

Thus, by Gerretsen's inequality $x_0 \geq 0$, it remains to show prove $J_0 > 0$ under Case 1. Note that J_0 can be rewritten as

$$\begin{aligned} J_0 =& 16e^3(32e^3 - 108e^2r - 395er^2 + 8218r^3) + 575316e^2r^4 \\ &+ 696816er^5 + 198288r^6, \end{aligned} \tag{3.27}$$

where $e = R - 2r \geq 0$. So, we only need to show that

$$32e^3 - 108e^2r - 395er^2 + 8218r^3 > 0.$$

To do this, we can assume that $r = 1$ and we have to show

$$32e^3 - 108e^2 - 395e + 8218 > 0, \tag{3.28}$$

which can be rewritten as

$$32(e^3 - 4e^2 - 15e + 64) + 20e^2 + 85e + 6170 > 0.$$

It remains to show

$$e^3 - 4e^2 - 15e + 64 > 0. \tag{3.29}$$

Putting $e = 3 + t$, then it becomes

$$t^3 + 5t^2 - 12t + 10 > 0.$$

Note that $5t^2 - 12t + 10 > 0$, thus the above inequality holds for $t > 0$ and then inequality (3.29) is proved when $e > 3$. On the other hand, inequality (3.29) can be rewritten as

$$(3 - e)(-e^2 + e + 18) + 10 > 0,$$

which is clear true for $0 < e < 3$. Therefore, we deduce that inequality (3.29) holds for any positive real number $e > 0$. And, inequality (3.25) is proved under Case 1.

Case 2. $I_0 \leq 0$.

Note that H_0 is a quadratic function (in s^2) with positive quadratic term. According to the property of quadratic functions with one variable, to prove $H_0 > 0$ under Case 2 we need to show $c_0 > 0$ and $F_i < 0$, where F_i is the discriminant of H_0 given by $F_i = b_0^2 - 4a_0c_0$.

By the assumption $I_0 \leq 0$ and Gerretsen's inequality (2.48) we have

$$\begin{aligned} &(104R^2 + 222Rr + 94r^2)(16Rr - 5r^2) - 400R^4 + 1504R^3r + 2444R^2r^2 \\ &- 2550Rr^3 - 2802r^4 \leq 0. \end{aligned}$$

Simplifying gives us

$$K_0 \equiv -400R^4 + 3168R^3r + 5476R^2r^2 - 2156Rr^3 - 3272r^4 \leq 0. \quad (3.30)$$

Through analysis, we obtain the following identity:

$$\begin{aligned}15625c_0 =& -K_0(20000R^2 + 100900Rr + 1682928r^2)\\&+ 2r^3(2915962152e^3 + 21562514576e^2r\\&+ 49054134321er^2 + 32643184221r^3),\end{aligned} \quad (3.31)$$

where $e = R - 2r \geq 0$. Since $K_0 \leq 0$, we have $c_0 > 0$.

On the other hand, it is easy to compute the discriminant F_i as follows:

$$\begin{aligned}F_i =& -52992R^8 + 285696R^7r - 4885632R^6r^2 - 10530560R^5r^3\\&+ 19072528R^4r^4 + 61834560R^3r^5 + 45018352R^2r^6\\&+ 633280Rr^7 - 4694976r^8.\end{aligned} \quad (3.32)$$

Also, it is easy to check the following identity:

$$9765625F_i = K_0L_0 - 64r^5M_0, \quad (3.33)$$

where

$$\begin{aligned}L_0 =&1293750000R^4 + 3271500000R^3r + 162899842500R^2r^2\\&+ 1585074025100Rr^3 + 14290030971992r^4,\\M_0 =&827889280426379e^3 + 6121431766559877e^2r\\&+ 13988526742361367er^2 + 9384579224935542r^3.\end{aligned}$$

Since $L_0 > 0$, $M_0 > 0$ and $K_0 \leq 0$, we deduce that $F_i < 0$ from identity (3.33) and inequality $I_0 > 0$ is proved under Case 2.

Combining the arguments of the above two cases, we proved that the strict inequality (3.25) holds for all triangles. Therefore, inequalities (3.20), (3.15), (3.12) and (1.11) are proved. It is easy to determine that the equality in (3.14) holds if and only if ABC is equilateral and we then further easily deduce that the equality in (1.11) holds if and only if triangle ABC is equilateral and P is its center. This completes the proof of Theorem 2.

4. Proof of Theorem 3

In this section, we shall prove Theorem 3. The following two lemmas will be used in the proof.

Lemma 10. *For an interior point P of triangle ABC the following inequality holds:*

$$aR_1 \geq br_3 + cr_2, \tag{4.1}$$

with equality if and only if P lies on the line AO (O is the circumcenter of triangle ABC).

We can find various proofs of inequality (4.1) from a lot of papers (see, for example [12]-[14]) related to the famous Erdös-Mordell inequality:

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3). \tag{4.2}$$

Lemma 11. *In any triangle ABC the following inequality holds:*

$$(m_b + m_c)^2 \geq \frac{9}{4}a^2 + h_a^2, \tag{4.3}$$

where h_a is the altitude from vertex A to the side BC. Equality in (4.3) holds if and only if $b = c$.

For a proof of inequality (4.3), see [15].

Next, we prove Theorem 3.

Proof

According to Lemma 10, to prove inequality (1.12) we need to show

$$\sum m_a^2 \frac{br_3 + cr_2}{ar_1} \geq 2\sum m_a^2,$$

which is equivalent to

$$\sum a\left(\frac{r_3}{br_2}m_b^2 + \frac{r_2}{cr_3}m_c^2\right) \geq 2\sum m_a^2.$$

Since

$$\frac{r_3}{br_2}m_b^2 + \frac{r_2}{cr_3}m_c^2 \geq \frac{2m_bm_c}{\sqrt{bc}},$$

We only need to prove that

$$\sum \frac{a}{\sqrt{bc}}m_bm_c \geq \sum m_a^2. \tag{4.4}$$

Note that $b+c \geq 2\sqrt{bc}$. We shall show the following stronger inequality:

$$\sum \frac{a}{b+c} m_b m_c \geq \frac{1}{2} \sum m_a^2. \tag{4.5}$$

It follows from Lemma 11 that

$$2m_b m_c \geq \frac{9}{4}a^2 + h_a^2 - m_b^2 - m_c^2.$$

Using the previous median formula (3.4) we further obtain

$$m_b m_c \geq \frac{1}{8}(5a^2 - b^2 - c^2 + 4h_a^2). \tag{4.6}$$

Thus, to prove inequality (4.5) we only need to prove

$$\sum \frac{a}{b+c}(5a^2 - b^2 - c^2 + 4h_a^2) \geq 3 \sum a^2. \tag{4.7}$$

Using the known formula:

$$h_a = \frac{\sqrt{2\sum b^2c^2 - \sum a^4}}{2a}, \tag{4.8}$$

we easily get

$$\begin{aligned} &\sum \frac{a}{b+c}(5a^2 - b^2 - c^2 + 4h_a^2) - 3\sum a^2 \\ =&\frac{N_0}{abc(b+c)(c+a)(a+b)}, \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} N_0 =& 2\sum b^4c^4 + 2abc\sum a^5 - 2(abc)^2 \sum bc - \sum a^2(b^6 + c^6) \\ &+ abc \sum a^2(b^3 + c^3) - abc \sum a(b^4 + c^4). \end{aligned}$$

It remains to prove that

$$N_0 \geq 0. \tag{4.10}$$

Putting $b+c-a = 2x, c+a-b = 2y, a+b-c = 2z$, the we have $a = y+z, b = z+x$ and $c = x+y$. It is easy to obtain the following

identity:

$$\begin{aligned}4N_0 = & \sum x(y^7+z^7)+\sum x^2(y^6+z^6)-\sum x^3(y^5+z^5)\\ & +2xyz\sum x(y^4+z^4)-6xyz\sum x^2(y^3+z^3)\\ & -2\sum y^4z^4+2(xyz)^2\sum yz. \end{aligned} \tag{4.11}$$

After analysis, we obtain the following identity:

$$\begin{aligned}8N_0 = & 4xyz\sum x\sum x^2(x-y)(x-z)\\ & +2\sum yz(x^2+4zx+4xy+y^2+z^2+yz)(y-z)^2(y+z-x)^2\\ & +xyz\sum[3x(y^2+z^2+4yz)+(y+z)(3x^2+5yz)](y-z)^2, \end{aligned} \tag{4.12}$$

Note that Schur's inequality (cf. [5, Theorem 80]):

$$\sum x^2(x-y)(x-z)\geq 0, \tag{4.13}$$

we deduce $N_0 \geq 0$ from (4.12). Therefore, inequalities (4.5) and (1.12) are proved. Moreover, it is easy to determine that equality in (1.12) holds if and only triangle ABC is equilateral and P is its center. This completes the proof of Theorem 3.

5. **Three conjectures**

In the last section, we propose three conjectures as open problems.

It is natural to consider unified exponential generalizations of inequality (1.6) and (1.12). After checking by the computer, we propose the following conjecture:

Conjecture 1. *Let k be a real number such that $-1 \leq k \leq 3$. Then for an interior point P of triangle ABC the following inequality holds:*

$$m_a^k\frac{R_1}{r_1}+m_b^k\frac{R_2}{r_2}+m_c^k\frac{R_3}{r_c}\geq 2(m_a^k+m_b^k+m_c^k). \tag{5.1}$$

For inequality (4.4), we propose the following exponential generalization:

Conjecture 2. *Let k be a real number such that $0 < k \leq \frac{3}{2}$. Then for any triangle ABC the following inequality holds:*

$$\frac{a}{\sqrt{bc}}(m_b m_c)^k + \frac{b}{\sqrt{ca}}(m_c m_a)^k + \frac{c}{\sqrt{ab}}(m_a m_b)^k \\ \geq m_a^{2k} + m_b^{2k} + m_c^{2k}. \tag{5.2}$$

If this conjecture is true, then from the proof of Theorem 1 we can easily deduce that inequality (5.1) holds when $0 < k \leq 3$.

For the acute triangle ABC, the author conjectures that inequality (1.11) can be improved to the following:

Conjecture 3. *For an interior point P of acute triangle ABC the following inequality holds:*

$$m_a \frac{R_1}{r_1} + m_b \frac{R_2}{r_2} + m_c \frac{R_3}{r_3} \geq 9R. \tag{5.3}$$

Remark 3. *The author has known that inequality* (5.3) *can not be proved by using the method to prove inequality* (1.11).

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